Problem Set 1: Consumption Theory & Labour Supply

Consumption Theory & Labour Supply

Exercise 1. Consider a one-period model where the only income source is labour. Suppose the utility function of the representative agent is $log(c) - \frac{l^2}{2}$, and there is a constant tax rate t on labour supply:

1. What is the optimal level of labour supply?
2. What revenue is raised (call this T)?
3. If a lump-sum tax of T was imposed on the worker, what would happen to labour supply? (Use intuition.)

Solution 1 (Labour Supply & Taxes)**.**

1. We do not need time subscripts since there is one period only. Normalising price to 1 so w is real wage, preferences are given by $U(c,l) = \log(c) - \frac{l^2}{2}$ and the household budget constraint is given by $c = (1-t)wl$. The Lagrangian is $u(c,l) + \lambda[(1-t)wl - c]$. First order conditions with respect to consumption and labour are $u'(c) = \lambda$ and $v'(l) = \lambda(1-t)w$, respectively and dividing the latter by the primer yields $(1-t)wu'(c) = v'(l)$ after multiplying across by $u'(c)$. Using the functional form for $u'(c)$ and $v'(l)$ we get $(1-t)w\frac{1}{c} = l$ and using the household budget constraint to substitute for c, we find that the optimal level of labour supply is one.
2. Total earnings are $(1-t)wl$ and government revenue will be $T = twl$. Since $l = 1$, $T = tw$.
3. A lump sum tax will not affect the labour/leisure choice so $wu'(c) = v'(l)$ so $w\frac{1}{c} = l$ but the household budget constraint will now be $c = wl - T$ so $w\frac{1}{wl-T} = l$. There is no substitution effect since the labour/leisure choice is undistorted. There will be only a pure income effect where agents feel poorer so reduce their consumption of leisure, i.e. labour supply will rise.

Exercise 2. Write down the labour-leisure optimality condition in terms of the real wage, $\frac{W}{P}$. Assuming people derive all their income from labour income, show that a sales tax has the same economic effects on labour supply as a tax on labour. Explain the intuition.

Solution 2 (Consumption versus labour tax equivalence on labour supply)**.** Without normalisation, Lagrangian is $u(c,l) + \lambda[Wl - Pc]$ so we will get $\frac{W}{P}u'(c) = v'(l)$. Assuming people derive all their income from labour income, household budget constraint is $Wl = Pc$. With a sales tax, household budget constraint is $Wl - (1+t)Pc$, while with labour income tax, the budget constraint is $(1-t)Wl - Pc$. These result in labour/leisure conditions of $v'(l) = \frac{1}{1+t}\frac{W}{P}u'(c)$ and $v'(l) = (1-t)\frac{W}{P}u'(c)$, respectively. In each case, t is inversely proportional to labour supply. With a tax on sales, the marginal gain from consumption is decreased because it is more expensive to consume so to reduce the marginal disutility from working, less labour is supplied. With a tax on labour, agents substitute away from working: the marginal gain via the extra consumption utility from marginal real wage is reduced so to reduce their disutility (pain) from working, they work less. However, like example 2 in lecture slides 1, there is both an income effect and a substitution effect. As to which dominates, long-run evidence suggests that income effect dominates, so labour supply would rise; note that this follows from figures 2.1-2.3 in the notes if you flip the situation: instead of rising real wages leading to declining labour hours, declining wages lead to increasing labour hours. If t rises, the budget constraint and the first-order conditions must be satisfied. 4 cases must be considered:

1. C rises and l falls: can't work less and consume more so this violates the budget constraint.
2. C rises and l rises: marginal disutility rises but marginal utility falls so this contradicts the labour/leisure condition.

3. C falls and l falls: possible.
4. C falls and l rises: possible.

So, C falls unambiguously, but we don't know if l falls or rises without knowing which effect (income or subsitution) dominates; what happens depends on the functional form of utility.

Exercise 3. Suppose we have a two period model, and each period utility is given by

$$u(C) = \frac{C^{1-\theta}}{1-\theta} - \frac{L^{1+\sigma}}{1+\sigma}$$

Assuming $\beta(1+r) = 1$ show that

$$\frac{l_1}{l_2} = \left(\frac{w_1}{w_2}\right)^{\frac{1}{\sigma}}$$

Explain the intuition (esp. the role of σ), and use the condition to explain what happened in Iceland in 1987.

Solution 3 (Elasticity of Intertemporal Substitution)**.** Euler equation is $u'(C_1) = \beta u'(C_2)(1+r)$ and labour/leisure condition is $w_t u'(C_t) = v'(l_t)$, $t \in \{1, 2\}$. Since $\beta(1+r) = 1$, $u'(C_1) = u'(C_2)$ so dividing the labour/leisure conditions for time one and two, we have $\frac{w_1}{w_2} = \frac{v'(l_1)}{v'(l_2)}$. Noting that $v'(l) = l^\sigma$, we have our result. (Can also derive this through Lagrangian with two period budget constraint: $W_t l_t + (1+r_{t-1})B_{t-1} = P_t C_t + B_t$). If the wage in the first period exceeds that in the second, agents will be tempted to substitute towards working today (elasticity of intertemporal substitution) rather than tomorrow and the extent to which they substitute labour supply intertemporally is governed by how small σ is (how big $\frac{1}{\sigma}$ is). σ controls the degree of marginal disutility to working: if working a little extra is not too hard, then $\frac{1}{\sigma}$ will be larger – they will substitute more work today for less in the future. The form of $v(l) = \frac{l^{1+\sigma}}{1+\sigma}$ is convex, so disutility of supplying labour increases in labour. Iceland's move from a tax system where taxes were paid on the previous year's income to a pay-as-you-go system resulted in an essentially tax-free 1987 year, so it is an example of a supply-side experiment of temporary declines in income tax. Equivalently, we could think of $\frac{w_1}{w_2}$ rising. The employment rate (ratio of total number of weeks worked to potential supply by all working age individuals) rose by about 3% and then went down to its earlier level in 1988. Female employment grew by about 4.16% and male employment grew by about 2.36%, so there was an overall rise in the labour supply. Since the 'reduction in income tax' was *temporary*, there would be less of an income effect and more of a substitution effect; hence, the experiment provided the upper bound on the labour supply response. Elasticities of intertemporal substition differed across individuals. See Bianchi, Gudmundsson & Zoega (2001) in AER, especially Figure 1. The fact that people take long summer vacations yet work five day weeks means that the convexity is not too strong and equivalently, the elasticity of intemporal substitution is significant, which the Iceland 1987 experiment would seem to show.

Exercise 4. Suppose the utility function is

$$u(c, l) = \log(c) - \gamma\frac{l^{1+\sigma}}{1+\sigma}$$

The real wage is w. Assume consumption is financed through labour income.

a) What role does σ play?

b) Find the optimal level of labour supply.

c) What is the interpretation of γ? Could differences in labour supply across countries be explained by differences in γ?

d) Suppose now utility takes the more general form $u(c,l) = \frac{c^{1-\theta}}{1-\theta} - \gamma\frac{l^{1+\sigma}}{1+\sigma}$ and $\theta = \frac{1}{2}$. Find the optimal level of labour supply. What happens to labour supply if the government imposes a proportional tax on the real wage?

e) Explain what happens to labour supply if the government gives people a "handout" of d. (No need for maths.)

Solution 4 (Labour Supply)**.**

a) σ controls the degree of marginal disutility to working: if working a little extra is not too hard, then $\frac{1}{\sigma}$ will be larger – they will substitute more work today for less in the future. The form of $v(l) = \frac{l^{1+\sigma}}{1+\sigma}$ is convex, so disutility of supplying labour increases in labour.

b) The static neoclassical labour/leisure optimality condition is $wu'(c) = v'(l)$ (can arrive via Lagrangian or arbitrage or law of equi-marginal returns. . .). Since consumption is financed through labour income $c = wl$ so plugging this in after taking derivatives of functional forms (so $\frac{w}{c} = l^\sigma$) we get that $\frac{1}{l} = l^\sigma$ so $l = 1$ is the optimal level of labour supply: work all the hours (since there is only one period and consumption is financed entirely through consumption, given the form of preferences, this makes sense).

c) γ is a taste parameter that governs the taste for work/leisure. It is distinct from σ, which mediates the disutility from *additional* work. Yes, preferences might account for the differences in labour supply across countries. The French, say, simply place greater value on leisure (or, equivalently, suffer greater disutility from labour). Retirees. Prime-age workers. Participation.

d) Now $u'(c) = c^{-1/2}$ so ICBST: $l = w^{\frac{1}{2\sigma+1}}$. With a proportional tax on wage, the return to working is less but there is no effect on consuming if you do not work so the substitution effect means we work less but there is no income effect. Working the maths out, the new budget constraint is $c = (1-t)l$ and the new labour/leisure optimality condition is $(1-t)wu'(c) = v'(l)$. ICBST: $l = [(1-t)w]^{\frac{1}{2\sigma+1}}$.

e) The handout of d will not affect the labour/leisure tradeoff, but will purely lead to an income effect since now $c = wl + d$ (i.e. consumption can be financed through labour income *plus* the "handout"). As people feel richer, they will work less (income effect). Since there is no substitution effect (labour/leisure tradeoff unaffected), there is a pure income effect, so people unambiguously work less when the government gives people a "handout" of d. Alternatively, plugging $c = wl + d$ into $u'(c)$, we know c now increases so by diminishing marginal utility, $u'(c)$ falls and so by arbitrage (labour/leisure tradeoff), $v'(l)$ (disutility to working) must fall and since this is convex, l must fall, i.e. optimal labour supply falls.

Exercise 5. Suppose lifetime utility is given by

$$u(C_1, l_1, C_2, l_2) = \log(C_1) - .5l_1^2 + \log(C_2) - .5l_2^2$$

The real interest rate is r and wages in periods 1 and 2 are w_1 and w_2, respectively. Write down the intertemporal budget constraint, assuming the government imposes a lump sum tax of T each period. Write down the first order conditions for labour and indicate how labour supply is affected by this change.

Solution 5 (Lump Sum Tax)**.** Intertemporal budget constraint:

$$C_1 + \frac{C_1}{1+r} + T + \frac{T}{1+r} = w_1 l_1 + \frac{w_2 l_2}{1+r}$$

Let λ be the Lagrangian multiplier. First order conditions:

$$\begin{aligned}
[C_1]: &\quad \frac{1}{C_1}\lambda \\
[C_2]: &\quad \frac{1}{C_2}\frac{\lambda}{1+r} \\
[l_1]: &\quad l_1 = \lambda w_1 \\
[l_2]: &\quad l_2 = \frac{\lambda w_2}{1+r}
\end{aligned}$$

So, $\frac{C_2}{C_1} = 1 + r$ and $\frac{l_1}{l_2} = (1+r)\frac{w_1}{w_2}$ or similarly $\frac{w_1}{c_1} = l_1$ and $\frac{w_2}{c_2} = l_2$. Because income falls, the consumer is poorer; as a result, demand for consumption and *leisure* will fall. For this reason, labour supply rises – a *pure income effect*. Can see this since pluggig in for c_1 and c_2 from intertemporal budget constraint into FOCs, c_1 and c_2 decline due to lump sum taxes in IBC and since c_1 and c_2 are on denominators, this means l_1 and l_2 rise, unambiguously. Relative prices are not distorted making this tax more efficient. E.g. labour supply of old during recession. Likewise, lottery winners reduce their labour supply. Another example of a lump-sum tax could be a property tax.

Exercise 6. Consider the *Ricardian* approach to fiscal policy, but now assume endogenous income (via labour supply). If government expenditure rose permanently, what would happen to i) consumption; ii) labour supply?; and iii) output? With "GHH" preferences, the marginal utility of consumption is increasing in the level of labour supply (i.e., consumption and labour are complements in utility). How would this change the results?

Solution 6 (Ricardian Equivalence)**.** Permanent changes strengthen the income effect over the substitution effect. With a permanent increase in G following Ricardian equivalence (T must increase permanently), people will feel poorer and by income effects reduce C and increase L supply. Output will rise. Can also see this from considering

$$\begin{aligned}
C_1 + \frac{C_2}{1+r} + G_1 + \frac{G_2}{1+r} &= w_1 l_1 + \frac{w_2 l_2}{1+r} \\
C_1 + \frac{C_2}{1+r} &= w_1 l_1 + \frac{w_2 l_2}{1+r} - G_1 - \frac{G_1}{1+r}
\end{aligned}$$

Due to the negative income effect, consumption will fall, labour supply will rise and output will rise. With GHH preferences, consumption and labour supply are complements, so consumption will rise relative to the first case.

Exercise 7. Suppose there are three tax brackets in a progressive tax system. The marginal tax rate is 10% up to $10,000$, 20% from $10,000$ to $20,000$, and 30% above $20,000$. Suppose the middle rate is reduced from 20% to 10%.

1. What effect will this have on the labour supply of a worker earning i) $10,000$; ii) $20,000$; and iii) $30,000$?
2. What is the maximum tax revenue from a worker earning $30,000$ now?
3. Explain why social welfare can be improved by reducing the marginal tax rate of the richest worker in the economy to zero. Does this logic hold for the *second* richest person in the economy?

Solution 7 (Marginal Tax Rates)**.**

1.

(a) A worker earning $10,000$ pays a 10% average tax rate and a 20% marginal tax rate before the change in the tax rates. Afterwards, there is no change to the average tax rate (no income effect) but the marginal tax rate falls to 10%, inducing a substitution effect and leading to a rise in labour supply.

(b) A worker earning $20,000$ pays a 15% average tax rate and a 30% marginal tax rate before the change in the tax rates. Afterwards, there is no change to the marginal tax rate (no substitution effect) but the average tax rate falls to 10%, inducing an income effect and leading to a fall in labour supply.

(c) A worker earning $30,000$ pays a 20% average tax rate and a 30% marginal tax rate before the change in the tax rates. Afterwards, there is no change to the marginal tax rate (no substitution effect) but the average tax rate falls to $16\frac{2}{3}\%$, inducing an income effect and leading to a fall in labour supply.

2. Worker on $20,000$ earns after tax income of $17,000$ ($18,000$ with new tax rates). Could tax worker on $30,000$ up to point where worker earns just a bit more than worker on $20,000$ after tax, so maximum tax revenue would be $13,000$ ($12,000$ with new tax rates) from a worker earning $30,000$.

3. Mirlees: say the richest worker in the economy earns \$500 million a year and pays \$100 million a year in taxes. This worker would be incentivised to supply more labour if any additional income was not taxed (marginal tax of zero). Since the government would not lose any revenue and there would be more labour supplied (voluntarily by the richest worker since s/he would be better off), one person gains (the richest worker) and no-one else is worse off, so we have found a Pareto improvement and can improve societal welfare. However, if the income gap between second richest worker is non-zero, setting the marginal rate of tax to zero at the second richest worker's income level would mean that the government loses the revenue it had from the richest worker over and above that of the second richest worker. The richest worker was already happy to work that much so didn't need to be incentivised through further tax cuts. The government is worse off and so we cannot have a Pareto improvement.

Exercise 8. In year 1, suppose that there is a temporary increase in government expenditure (due, say, to a war) in country A. Meanwhile, in country B there is a permanent rise in expenditure due to an expansion of the welfare state. According to the theory of *tax smoothing*, what happens to the government's *bond issuance* in each country in year 1?

Solution 8 (Tax Smoothing & Persistence of G)**.** Since the increase in government expenditure in country A is *temporary*, country A should issue debt to finance its expenditure, according to the theory of tax smoothing because high tax rates are distortionary. In country B, the *permanent* nature of the government expenditure shock should be met by a *permanent* increase in taxes.

Exercise 9. Write down the undistorted labour/leisure in terms of the real wage, $\frac{W}{P}$, and assume optimal labour supply is l^*. Suppose now there is a proportional sales tax of rate τ on all goods. To ensure labour supply of l^*, at what rate would the government have to *subsidise* nominal wages?

Solution 9 (Tax Smoothing & Persistence of G)**.** (This is similar to Question 1 of Problem Set 1). Without normalisation, Lagrangian is $u(c,l)+\lambda[Wl-Pc]$ so we will get that the undistorted labour/leisure condition in terms of the real wage is $v'(l)=u'(c)\frac{W}{P}$. With a proportional sales tax τ, the household budget constraint is $Wl-(1+\tau)Pc$ so the labour/leisure condition becomes $v'(l)=\frac{1}{1+\tau}\frac{W}{P}u'(c)$. With a government subsidy for the nominal wage of ξ, the houshold gets $(1+\xi)Wl$ in nominal terms for supplying l hours and the labour/leisure condition becomes $v'(l)=\frac{1+\xi}{1+\tau}\frac{W}{P}u'(c)$. Since l^* was the optimal labour in the undistorted case, we need here that $\frac{1+\xi}{1+\tau}\frac{W}{P}u'(c)=\frac{W}{P}u'(c)\implies\xi=\tau$. So, to ensure labour supply of l^*, the government would have to subsidise nominal wages at rate $\xi=\tau$, the same rate of proportional sales tax.

Exercise 10. Multiple Choice Questions: 2012 Midterm, Questions 1-7 & 12.

Solution 10 (MCQs)**.** a, a, b, b, a, c, c, d.