Health Warning

Use these slides in combination with (not in isolation from) SEMnotes.pdf.

Simultaneous Equations Models

- SEMs describe an interrelated system of relationships.
- Simple equation model:

  $$Y = f(X, u) \overset{e.g.}{=} \beta_0 + \beta_1 X + u$$

  X fixed / independent so $Cov(X_{ji}, u_i) = 0$ for all i, i.e. X exogenous.

  One way causation – see graph.
- Strict exogeneity – lack of correlation – any omitted variable (in u) cannot be correlated with X.
- Does not hold in most econometric applications – equation cannot exist in interrelated, dynamic system.
- Without lagged variables, there can be no dynamics.
- Right-hand side lagged variables determined by u in last period.
- Endogenous $X \rightarrow$ two way causation, i.e. joint dependency / simultaneity so $Cov(X_{ji}, u_i) \neq 0$ violating Classical assumptions; see graph.

Examples of Simultaneity

Competitive Market

Q affects P and P affects Q:

$$Q_D = \alpha + \beta P + u \tag{1}$$

$$Q_S = \gamma + \delta P + v \tag{2}$$

$$Q_D = Q_S \tag{3}$$

Solving for the one unknown, P:

$$\begin{aligned}
\alpha + \beta P + u &= \gamma + \delta P + v \\
(\beta - \delta)P &= \gamma - \alpha + v - u \\
P &= \frac{\gamma - \alpha}{\beta - \delta} + \frac{v}{\beta - \delta} - \frac{u}{\beta - \delta} \\
&= \pi + \frac{v - u}{\beta - \delta}
\end{aligned}$$

$Cov(P, u) \neq 0$ and $Cov(P, v) \neq 0$ since P depends on u and v, i.e. P is correlated with the demand shock and the supply shock.

Examples of Simultaneity

Competitive Market

Calculating measures for these dependencies:

$$
\begin{aligned}
Cov(P,u) &= E(P-E(P))(u-E(u)) \\
&= E\left[\pi + \frac{v-u}{\beta-\delta} - \pi\right][u-0] \\
&= E\left[\left(\frac{v-u}{\beta-\delta}\right)u\right] \\
&= E\left(\frac{vu}{\beta-\delta}\right) - E\left(\frac{u^2}{\beta-\delta}\right) \\
&= \frac{-\sigma_u^2}{\beta-\delta} \neq 0
\end{aligned}
$$

- Typically, $\beta < 0$, i.e. higher prices reduce Q_D and $\delta > 0$, i.e. higher prices increase Q_S.
- $Cov(P,u) > 0$, i.e. positive demand shock raises price of a normal good; hence, price and demand shocks move in same direction.

Examples of Simultaneity

Simple Macro Model

$$Y \equiv C + I + G + X - M \quad \ldots Y \text{ endogenous}$$

$$\begin{aligned} C &= a_0 + a_1 Y + u \\ I &= b_0 + b_1 Y + b_2 r + v \\ Y &= C + I + G \end{aligned}$$

- $Cov(Y, u) \neq 0$ and $Cov(Y, v) \neq 0$
- Classical assumptions violated.

Structural Form of SEM

Terminology: Definitions

1. Variable types:
   - Endogenous:
   - Exogenous:
   - Predetermined: exogenous and lagged endogenous.
2. Types of relationships:
   - Behavioural:
   - Technological:
   - Identities:
   - Equilibrium condition:
3. Completeness: number of equations is equal to number of endogenous variables – allows us solve SEM uniquely.

Structural Form of SEM

Terminology: Example

$$\text{Behavioural } \begin{cases} Q_D = \alpha + \beta P + u \\ Q_S = \gamma + \delta P + v \end{cases}$$

$$\text{Equilibrium condition } Q_D = Q_S$$

- 4 structural parameters: (α, β, γ, δ).
- 1 exogenous variable: 1.
- 3 endogenous variables: Q_D, Q_S and P.
- Complete: since there are 3 equations and 3 endogenous variables.

Reduced Form of SEM

- Reduced form: endogenous variables expressed in terms of predetermined variables.
- Endogenous variables on the left-hand side and only predetermined on the right-hand side.
- Micro example:

$$\begin{aligned} P &= \frac{\gamma - \alpha}{\beta - \delta} + \frac{v}{\beta - \delta} - \frac{u}{\beta - \delta} \\ &= \pi + \frac{v - u}{\beta - \delta} \\ &= \pi_1 + w_1 \end{aligned}$$

$$\begin{aligned} Q &= \frac{\beta\gamma - \alpha\delta}{\beta - \delta} + \frac{\beta v - \delta u}{\beta - \delta} \\ &= \pi_2 + w_2 \end{aligned}$$

Identification

Intuition

$$Q_D = \alpha + \beta P + u$$
$$Q_S = \gamma + \delta P + v$$
$$Q_D = Q_S$$

- Identification: can we meaningfully estimate the structural parameters?
- Yes: equation is identifiable.
- No: equation is not identifiable.
- Two ways to view problem of identification:
  1. Mathematical.
  2. Statistical.

Identification

Intuition: Mathematical

- Goal: α, β, γ and δ.
- OLS estimation of reduced form:

$$P = \pi_1 + w_1 = \frac{\gamma - \alpha}{\beta - \delta} + w_1$$

$$Q_D = Q_S = \pi_2 + w_2 = \frac{\beta\gamma - \alpha\delta}{\beta - \delta} + w_2$$

- 2 equations in 4 unknowns – cannot solve uniquely for 4 structural parameters given 2 reduced form parameters – i.e. structural parameters are not identified – so supply and demand functions are unidentifiable.

Identification

Intuition: Mathematical

- For complete SEM:
  M: endogenous variables and equations.
  K: predetermined variables.
- Up to $M^2 - M$ structural parameters on endogenous variables and up to MK structural parameters on predetermined variables.
- Only up to MK reduced form parameters – so only up to MK equations relating structural and reduced form parameters.
- Impossible in general to go from knowledge of reduced form to knowledge of structural form parameters.
- If $M^2 - M > MK$ and more structural parameters than reduced form parameters, no meaningful estimation possible since we cannot estimate more than MK parameters in structural form with only MK equations.
- Model is solvable, but solution will not be unique.

Identification

Intuition: Statistical

- Graphs: demand and supply functions look same – statistically indistinguishable from any arbitrary function of P and Q.
- Let $Q_D = Q_S = Q$ and k, c be arbitrary real numbers:

$$
\begin{aligned}
kQ &= k\alpha + k\beta P + ku \\
cQ &= c\gamma + c\delta P + cv \\
\therefore (k+c)Q &= k\alpha + c\gamma + (k\beta + c\delta)P + ku + cv \\
Q &= \frac{k\alpha + c\gamma}{k+c} + \left(\frac{k\beta + c\delta}{k+c}\right)P + \frac{ku + cv}{k+c} \\
Q &= A + BP + W
\end{aligned}
$$

- Observational equivalence: infinite number of functions observationally equivalent to demand and supply function and have same reduced form.
- Can estimate reduced form, but do not know if we can get back to a unique structural form.

Identification

Solution

- Intuition:
  - Mathematical approach: restrict number of structural parameters to equal the number of reduced form parameters.
  - Statistical approach: require equation to be stable while others vary so we can identify the stable equation.
- Formal rules:

  M: number of endogenous variables / equations in the model (complete model).

  m: number of endogenous variables in equation of interest.

  K: number of predetermined variables in model.

  k: number of predetermined variables in equation of interest.
- Order condition (necessary):

$$K - k \geq m - 1$$

$$\text{OR} \quad M + K - (m + k) \geq M - 1$$

- Under-identification... Just/exact identification... Over-identification...

Identification
Example

$$\begin{aligned} \text{Demand:} \quad & Q_D = \alpha + \beta_1 P + \beta_2 Y + u \\ \text{Supply:} \quad & Q_S = \gamma + \delta P + v \\ \text{Equilibrium:} \quad & Q_D = Q_S \end{aligned}$$

$$\begin{array}{cccccc} Q_D & +0 & -\alpha & -\beta_1 P & -\beta_2 Y & = u \\ 0 & +Q_S & -\gamma & -\delta P & +0 & = v \\ Q_D & -Q_S & +0 & +0 & +0 & = 0 \end{array}$$

- $M = 3$: $(Q_D,\ Q_S,\ P)$ and $K = 2$: $(1,\ Y)$.
- Demand equation: $m = 2$ $(Q_D,\ P)$ and $k = 2$ $(1,\ Y)$.
  $K - k = 2 - 2 = 0$ and $m - 1 = 2 - 1 = 1$.
  $K - k < m = 1$ so demand function is 'not' identified.
- Supply equation: $m = 2$ $(Q_S,\ P)$ and $k = 1$ (constant).
  $K - k = 2 - 1 = 1$ and $m - 1 = 2 - 1 = 1$.
  $K - k \geq m - 1$ so supply function 'may' be identified – 'just' identified if so.

Identification

- Rank condition (necessary and sufficient): $\rho(\Lambda) = M - 1$.
- Previous example:
  - For the demand function:

    $$\Lambda_D = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

    $\rho(\Lambda_D) = 1 \neq M - 1 = 2$ so demand function is 'not' identified.
  - For the supply function:

    $$\Lambda_S = \begin{bmatrix} 1 & -\beta_2 \\ 1 & 0 \end{bmatrix}$$

    $\rho(\Lambda_S) = 2 = M - 1$ so supply function is identified – 'just' identified.

Estimation

- OLS leads to simultaneity bias.
- Single equation methods to estimate SEM include OLS, Indirect Least Squares (ILS) and 2SLS.
- System methods include 3SLS and FIML.

Indirect Least Squares Estimation

- Appropriate for 'exact' identification, ILS estimates structural parameters from reduced form parameters.
- Works since number of reduced form parameters (OLS) is same as structural form parameters.
- Underidentified case:

$$\begin{aligned}
&\textit{Demand:} \quad && Q_D = \alpha + \beta_1 P + \beta_2 Y + u \\
&\textit{Supply:} && Q_S = \gamma + \delta P + v \\
&\textit{Equilibrium:} && Q_D = Q_S = Q
\end{aligned}$$

$$\begin{aligned}
P &= \pi_{11} + \pi_{12} Y + w_1 \\
Q &= \pi_{21} + \pi_{22} Y + w_2
\end{aligned}$$

4 reduced form parameters but 5 structural form parameters – non-unique solution.

Indirect Least Squares Estimation

- Just identified case:

$$\begin{aligned} &\text{Demand:} && Q_D = \alpha + \beta_1 P + \beta_2 Y + u \\ &\text{Supply:} && Q_S = \gamma + \delta_1 P + \delta_2 W + v \\ &\text{Equilibrium:} && Q_D = Q_S = Q \end{aligned}$$

$$\begin{aligned} P &= \pi_{11} + \pi_{12} Y + \pi_{13} W + w_1 \\ Q &= \pi_{21} + \pi_{22} Y + \pi_{23} W + w_2 \end{aligned}$$

6 reduced form parameters and 6 structural form parameters – unique solution possible.

Indirect Least Squares Estimation

- Over-identified case:

$$
\begin{aligned}
\text{Demand:} &\quad Q_D = \alpha + \beta_1 P + \beta_2 Y + u \\
\text{Supply:} &\quad Q_S = \gamma + \delta_1 P + \delta_2 W + \delta_3 C + v \\
\text{Equilibrium:} &\quad Q_D = Q_S = Q
\end{aligned}
$$

$$
\begin{aligned}
P &= \pi_{11} + \pi_{12} Y + \pi_{13} W + \pi_{14} C + w_1 \\
Q &= \pi_{21} + \pi_{22} Y + \pi_{23} W + \pi_{24} C + w_2
\end{aligned}
$$

8 reduced form parameters but 7 structural form parameters – non-uniqueness.

Two Stage Least Squares Estimation

- OLS leads to simultaneity bias, ILS only works for just-identified case but 2SLS appropriate also for over-identified case.
  1. Estimate reduced form equations for endogenous variables on right hand side of structural form equations of interest via OLS and obtain reduced form estimates of these endogenous variables.
  2. Substitute for right-hand side endogenous variables in structural equation using reduced form estimates from stage 1 as proxies (instruments) and estimate structural form equation by OLS.

Example

$$\begin{aligned}
&\text{Demand:} && Q_D = \alpha + \beta_1 P + \beta_2 Y + u \\
&\text{Supply:} && Q_S = \gamma + \delta_1 P + \gamma_2 W + \delta_3 C + v \\
&\text{Equilibrium:} && Q_D = Q_S = Q
\end{aligned}$$

Estimate over-identified demand equations as follows:

1. Estimate reduced form equation: $P = \pi_{11} + \pi_{12}Y + \pi_{13}W + \pi_{14}C + w_1$ via OLS and obtain $\hat{P} = \hat{\pi}_{11} + \hat{\pi}_{12}Y + \hat{\pi}_{13}W + \hat{\pi}_{14}C$ where $\hat{P}$ is a function of only predetermined variables.
2. Estimate modified demand equations $Q_D = \alpha + \beta_1 \hat{P}_1 + \beta_2 Y + u$ via OLS. This produces 2SLS estimates of structural form parameters.

Statistical Properties of 2SLS

- 2SLS biased but consistent.
- 2SLS asymtotically normal and asymtotically efficient.
- 2SLS preferred to OLS.
- Efficient single stage algorithms used in practice.
- Problem for large systems. . .
- 2SLS same as ILS in just identified case.
- 2SLS implementable in Stata.